

Frame-like Actions for Massless Mixed-Symmetry Fields in Minkowski space

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Abstract

A frame-like action for arbitrary mixed-symmetry bosonic massless fields in Minkowski space is constructed. The action is given in a simple form and consists of two terms for a field of any spin. The fields and gauge parameters are certain tensor-valued differential forms. The formulation is based on the unfolded form of equations for mixed-symmetry fields.

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Introduction

Due to the ever-growing interest in field theories in higher dimensions, the study of mixed-symmetry fields, i.e., those that correspond generally to neither symmetric nor antisymmetric tensor representations of the Wigner little algebra, was started from the paper [1] and continued further in [2, 3, 4, 5, 6].

In $4d$ the spin is a single (half)integer. In the general case the spin degrees of freedom are in one-to-one correspondence with finite-dimensional unitary representations of the Wigner little algebra, which is $\mathfrak{so}(d-2)^1$ for massless fields and $\mathfrak{so}(d-1)$ for massive ones. Therefore, the spin can be naturally characterized by Young diagrams. The most developed cases correspond to one-row (symmetric) and one-column (antisymmetric) diagrams (tensors), in the former case the $4d$ -spin is equal to the number n of boxes in one-row Young diagram ($n+1/2$ for fermions).

The theory of massless mixed-symmetry fields has been developing over the last decades within different approaches, which can be split into *light-cone*, *metric-like* and *frame-like* ones², in accordance with the types of fields used.

Within the *light-cone* approach one deals with tensors of the Wigner little algebra and, hence, manifest Lorentz symmetry is lost. Nevertheless, the *light-cone* approach turned out to be very effective in constructing cubic vertices, which is the first nontrivial attempt of introducing interactions. For instance, the cubic vertices of mixed-symmetry fields for $d \geq 6$ were constructed in [9, 10] (see [11] for a review), inspiring the investigation to find full non-linear theories of mixed-symmetry fields. The first attempts towards an explicit construction of manifestly covariant cubic vertices for mixed-symmetry gauge fields were made in [12, 13].

Within the *metric-like* approach fields are the world tensors, which are analogous to metric field $g_{\mu\nu}$. The *metric-like* approach can be split further into *constrained* and *unconstrained* approaches, according to whether or not some trace constraints are imposed on fields and gauge parameters³.

Within the *unconstrained* approach of [15, 16, 17, 18, 19] fields and gauge parameters are not subjected to any trace constraints. A nice feature of the *unconstrained* approach is its relation to the low-tension limit of free string field theory [16, 20, 21, 19, 22]. However, the actions for unconstrained mixed-symmetry fields are only available in a nonlocal form [14]. Also, within the BRST approach, in which no off-shell trace constraints are imposed, the fields with the spin corresponding to two-row Young diagrams were studied in detail [23, 24, 25, 26].

The *constrained*⁴ approach is ascribed to Fronsdal who showed in [28] that totally symmetric massless fields have to be subjected to double-trace constraints for the equations to be gauge invariant and to describe the correct number of physical degrees of freedom [29]. In the pioneering paper by Labastida [6] on massless mixed-

¹Strictly speaking the stability subalgebra of a light-like momentum is $\mathfrak{iso}(d-2)$, however $\mathfrak{iso}(d-2)$ -translations have to be realized trivially for massless fields, reducing $\mathfrak{iso}(d-2)$ to $\mathfrak{so}(d-2)$.

²Let us also mention the ambient approach of [7], see [8] for recent developments.

³For the review of all approaches see [14].

⁴In principle, fields can be taken to be irreducible Lorentz tensors, though the gauge parameters in addition to being algebraically irreducible must satisfy differential constraints [27].

symmetry fields, a set of generalized trace constraints for fields and gauge parameters was suggested, the gauge invariant equations were derived and the local action was constructed. Unfortunately, it was not proved in [6] that the correct number of physical degrees of freedom propagates, this was proved in [14].

Making use of Fock space oscillators allowed Labastida to write down constraints/gauge transformations/equations/action for all mixed-symmetry fields at once. However, due to the lack of Young symmetry constraints a Fock space vector contains in its decomposition multiple copies of the same representations. In order to single out a particular mixed-symmetry field Young symmetry projectors are required. For individual mixed-symmetry fields the constraints/the field equations/the gauge transformations were obtained in [14]. To single out the action for a particular mixed-symmetry field from [6] appears to be a very involved procedure.

Within the *frame-like* approach [30] fields are tensor-valued differential forms, which are generalizations of vielbein e_μ^a and Lorentz spin-connection $\omega_\mu^{a,b}$. The *frame-like* approach has a lot of advantages: the use of differential forms simplifies introducing interactions with gravity; the forms take values in certain irreducible representations⁵ of the Lorentz [34, 35, 36] or (anti)-de Sitter algebra [37, 38, 39, 40, 41], i.e., tangent tensors are not subjected to complicated double-trace constraints; frame-like fields can be recognized as Yang-Mills connections of the space-time symmetry algebra; the most important is that frame-like fields can be embedded into the full set of fields of the *unfolded* approach. From this perspective the *frame-like* approach is a very promising one.

The *unfolded* approach itself is a reformulation of field equations in the first order form with the help of exterior differential and differential forms [42, 43, 44]. The *unfolded* approach appears to have a very rich underlying structure, so-called Free Differential Algebras [45], which are also used in supergravity and M-theory [46, 47, 48, 49, 50].

Among arbitrary-symmetry fields, distinguished is a subclass of totally-symmetric massless higher-spin fields, which has been the most extensively studied. Free totally-symmetric higher-spin fields were reformulated within the *unfolded* approach [42] and then the consistent nonlinear deformation of the unfolded equations was found in $4d$ [51, 52], generalized further to arbitrary space-time dimension in [53].

The crucial ingredient is to find a higher-spin algebra, which is a non-abelian infinite-dimensional extension of the space-time symmetry algebra satisfying admissibility condition [54], i.e. a gauging of the algebra must match some unitary representation thereof. The admissibility condition relates group- and field-theoretical aspects of the theory, stating that in order for some gauge theory with gauge algebra \mathfrak{g} to be consistent the spectrum of relativistic fields that live on the solutions of the equations of motion has to match some unitary representation of \mathfrak{g} .

As for totally-symmetric higher-spin fields, the admissibility condition for higher-spin algebras turned out to be very restrictive, discarding most of possible higher-

⁵The forms that take values in reducible representation of the Lorentz or (anti)-de Sitter algebra describe reducible sets of massless fields [31], which are related to the tensionless limit of string theory [3, 32, 33, 16, 21]. This possibility can be referred to as the *unconstrained frame-like* approach.

spin multiplets. For example, the simplest higher-spin algebra contains a single copy of a field of each spin $s = 0, 1, 2, \dots$. For this reason, until admissible higher-spin algebras for mixed-symmetry fields are known it would be better to work with individual fields rather than fix any particular infinite multiplet.

Recently the unfolded form of equations for arbitrary mixed-symmetry bosonic and fermionic massless fields in the Minkowski space has been constructed [55]. It turned out that the rather unnatural algebraic constraints that must be imposed both on fields and gauge parameters within the *constrained* approach of Labastida can be easily explained. The Labastida fields and gauge parameters can be identified with certain components of tensor-valued differential forms, connections of the space-time symmetry algebra. It is worth stressing again that the *unfolded* approach succeeded in constructing full non-linear equations for totally-symmetric higher-spin fields.

In this letter we construct a simple *frame-like* action for arbitrary-spin massless mixed-symmetry fields in Minkowski space. In contrast to the *constrained* approach, where the number of terms in the action grows rapidly with the rank of a tensor, within the *frame-like* approach the action consists only of two terms for an arbitrary mixed-symmetry field; the Lagrangian equations are manifestly gauge invariant.

In Section 1 we recall the general properties of massless mixed-symmetry fields in Minkowski space, essential features of the *unfolded* approach are given in Section 2. The action is constructed in Section 3.

Notation

Greek indices $\mu, \nu, \dots = 0 \dots (d-1)$ are the world indices of the Minkowski space-time. Lowercase Latin letters $a, b, c, \dots = 0 \dots (d-1)$ are the tangent indices in the basis that is defined by background vielbein h_μ^a . $h_\mu^a h_\nu^b \eta_{ab} = g_{\mu\nu}$, $g_{\mu\nu}$ is a metric tensor in some coordinates and $\eta_{ab} = \text{diag}(1, -1, \dots, -1)$ is the invariant tensor of $\mathfrak{so}(d-1, 1)$. In Cartesian coordinates with $h_\mu^a = \delta_\mu^a$ there is no distinction between world and tangent tensors.

A group of n (anti)symmetric indices $a_1 \dots a_n$ or $ab \dots c$ is denoted $(a[n])$ $a(n)$ or $([ab \dots c])$ $(ab \dots c)$. (Anti)symmetrization is denoted by placing a group of indices in (square)round brackets, or by designating indices by the same letter. Only necessary permutations are performed⁶, e.g., for a vector V^a and a symmetric rank-two tensor $U^{ab} = U^{ba}$ $V^{(a} U^{bc)} \equiv V^a U^{bc} + V^c U^{ab} + V^b U^{ca}$ and $V^a U^{aa} \equiv V^{a_1} U^{a_2 a_3} + V^{a_2} U^{a_3 a_1} + V^{a_3} U^{a_1 a_2}$.

All the necessary information on Young diagrams and mixed-symmetry tensors is collected in Appendix A.

⁶Note that the (anti)symmetrization operation defined in this way is just an abridged notation for a number of distinct terms with permuted indices rather than a well-behaved projector.

1 Massless Mixed-Symmetry Fields

Though we work with world tensors in this Section, in order to simplify the comparison with the *unfolded* approach, it is useful to convert all world tensor indices to tangent ones by virtue of inverse vielbein $h^{\mu a}$, e.g., $\partial^a \equiv h^{\mu a} \partial_\mu$, $\phi^{a(s)} \equiv \phi_{\mu_1 \dots \mu_s} h^{\mu_1 a} \dots h^{\mu_s a}$.

We start by briefly reviewing the general properties of totally-symmetric and totally-antisymmetric fields, whose spin degrees of freedom are characterized⁷ by $\mathbf{Y}\{(s, 1)\}$ and $\mathbf{Y}\{(1, p)\}$ irreducible representations of the Wigner massless little algebra $\mathfrak{so}(d-2)$, respectively.

A symmetric spin- s field. As was found by Fronsdal in [28], a totally-symmetric spin- s massless field can be described⁸ by a symmetric rank- s tensor field $\phi^{(a_1 \dots a_s)}$, satisfying

$$\begin{aligned} \square \phi^{a_1 \dots a_s} - \partial^{(a_1} \partial_m \phi^{m a_2 \dots a_s)} + \partial^{(a_1} \partial^{a_2} \phi^{a_3 \dots a_s) m}{}_m &= 0, \quad \phi^m{}_m{}^{a_5 \dots a_s} \equiv 0, \\ \delta \phi^{a_1 \dots a_s} &= \partial^{(a_1} \xi^{a_2 \dots a_s)}, \quad \xi^m{}_m{}^{a_3 \dots a_{s-1}} \equiv 0, \end{aligned} \quad (1.1)$$

where in order for gauge parameter $\xi^{a_1 \dots a_{s-1}}$ to be differentially unconstrained only the second trace of the field has to vanish, i.e., the field has to take values in a reducible representation of $\mathfrak{so}(d-1, 1)$.

An antisymmetric (p -form) field. Analogously, a totally-antisymmetric massless field or p -form can be described by an antisymmetric rank- p tensor field $\omega^{[c_1 \dots c_p]}$, satisfying

$$\square \omega^{c_1 \dots c_p} - \partial^{[c_1} \partial_m \omega^{m c_2 \dots c_p]} = 0, \quad \delta \omega^{c_1 \dots c_p} = \partial^{[c_1} \xi^{c_2 \dots c_p]}, \quad (1.2)$$

where gauge parameter $\xi^{c_1 \dots c_{p-1}}$ is a rank- $(p-1)$ antisymmetric tensor. The gauge symmetry is reducible in the sense that not all of the gauge parameters do affect the field, these are represented by the second level gauge parameters $\xi^{c_1 \dots c_{p-2}}$, $\delta \xi^{c_1 \dots c_{p-1}} = \partial^{[c_1} \xi^{c_2 \dots c_{p-1}]}$ modulo those components of $\xi^{c_1 \dots c_{p-2}}$ that do not affect $\xi^{c_1 \dots c_{p-1}}$ and so on until $\delta \xi^c = \partial^c \xi$.

Mixed-symmetry massless fields join together nontrivial trace constraints of symmetric fields with reducible gauge symmetries of p -form fields and introduce a new feature of having more than one gauge parameter. Let us consider the case of a spin- $\mathbf{Y}\{s, t\}$, i.e., spin- $\boxed{t \quad s}$, field in detail and review general properties of arbitrary-spin mixed-symmetry fields.

⁷The diagrams are given by specifying the lengths of the rows as $\mathbf{Y}\{s_1, \dots, s_n\}$ or by specifying the widths and the heights of its rectangular subblocks, e.g., $\mathbf{Y}\{(s, 1)\} \equiv \mathbf{Y}\{s\}$ and $\mathbf{Y}\{(1, p)\} \equiv \mathbf{Y}\{\underbrace{1, \dots, 1}_p\}$. See Appendix A for more detail.

⁸In fact, there is no strong interdependence between the representation \mathbf{Y} of $\mathfrak{so}(d-2)$, i.e., the spin, and the representation of the Lorentz algebra \mathbf{Y}_M in which the field $\phi(x)$ takes values. The most important and the most natural choice is when $\mathbf{Y} = \mathbf{Y}_M$ as Young diagrams, all other are referred to as dual formulations. Note, that \mathbf{Y} corresponds to an irreducible representation of $\mathfrak{so}(d-2)$, whereas \mathbf{Y}_M correspond to a representation of $\mathfrak{so}(d-1, 1)$ that has to be reducible in most cases in order for equations of motion to be invariant under gauge transformations with differentially unconstrained parameters and for these field equations to admit a Lagrangian. For the minimal choice the term 'spin- \mathbf{Y} field' is unambiguous.

A spin-Y $\{s, t\}$ **field.** A spin-Y $\{s, t\}$ massless field can be described [6] by a field $\phi^{a(s), b(t)} \equiv \phi^{a_1 \dots a_s, b_1 \dots b_t}$, which is symmetric in $a_1 \dots a_s$ and $b_1 \dots b_t$, separately, and $\phi^{(a_1 \dots a_s, a_{s+1}) b_2 \dots b_t} \equiv 0$, i.e., $\phi^{a(s), b(t)}$ has the symmetry of $\begin{array}{|c|} \hline s \\ \hline t \\ \hline \end{array}$. Analogously to symmetric fields, $\phi^{a(s), b(t)}$ is subjected to double-trace constraints with respect to each of the two groups of symmetric indices

$$\phi^{m \ n \ a_5 \dots a_s, b_1 \dots b_t} \equiv 0, \quad \phi^{a_1 \dots a_s, m \ n \ b_5 \dots b_t} \equiv 0, \quad (1.3)$$

the cross traces with respect to a pair of indices from distinct groups need not vanish. Therefore, $\phi^{a(s), b(t)}$ contains a lot of irreducible components in general. The Labastida equations [6]

$$\begin{aligned} & \square \phi^{a(s), b(t)} - \partial^a \partial_c \phi^{a(s-1)c, b(t)} - \partial^b \partial_c \phi^{a(s), b(t-1)c} + \\ & + \partial^a \partial^b \phi^{a(s-1)c, b(t-1)}_c + \partial^a \partial^a \phi^{a(s-2)c, b(t)}_c + \partial^b \partial^b \phi^{a(s), b(t-2)c}_c = 0, \end{aligned} \quad (1.4)$$

share the symmetry and the trace properties of $\phi^{a(s), b(t)}$. The equations are invariant under gauge transformations⁹

$$\delta \phi^{a(s), b(t)} = \frac{s-t}{s-t+1} \partial^a \xi_1^{a(s-1), b(t)} - \partial^b \xi_2^{a(s), b(t-1)} + \frac{1}{s-t+1} \partial^a \xi_2^{a(s-1)b, b(t-1)}, \quad (1.5)$$

where gauge parameters $\xi_1^{a(s-1), b(t)}$ and $\xi_2^{a(s), b(t-1)}$ have the symmetry of $\begin{array}{|c|} \hline s-1 \\ \hline t \\ \hline \end{array}$ and $\begin{array}{|c|} \hline s \\ \hline t-1 \\ \hline \end{array}$, respectively, and

$$\xi_1^{a(s-3)m, b(t)}_m + \frac{2}{(s-t-1)} \left(\xi_1^{a(s-3)bm, b(t-1)}_m + \frac{2}{(s-t)} \xi_1^{a(s-3)bb, m \ b(t-2)}_m \right) \equiv 0, \quad (1.6)$$

$$\xi_2^{a(s), m \ b(t-3)}_m \equiv 0, \quad (1.7)$$

$$\text{any double trace of } \xi_1^{a(s-1), b(t)} \text{ and } \xi_2^{a(s), b(t-1)} \text{ vanishes,} \quad (1.8)$$

$$\begin{aligned} & \frac{2(s-t)}{(s-t-1)} \left(\xi_1^{a(s-2)m, b(t-1)}_m + \frac{1}{(s-t+1)} \xi_1^{a(s-2)b, m \ b(t-2)}_m \right) \equiv \xi_2^{a(s-2)m, b(t-1)}_m + \\ & + \frac{2}{(s-t+1)} \left(\xi_2^{a(s-2)bm, b(t-2)}_m + \frac{2}{(s-t+2)} \xi_2^{a(s-2)bb, m \ b(t-3)}_m \right), \end{aligned} \quad (1.9)$$

$$\xi_1^{a(s-1), m \ b(t-2)}_m \equiv 2 \xi_2^{a(s-1)m, b(t-1)}_m. \quad (1.10)$$

Identities (1.6) and (1.7) imply that the trace of $\xi_1^{a(s-1), b(t)}$ with the symmetry of $\begin{array}{|c|} \hline s-3 \\ \hline t \\ \hline \end{array}$ and the trace of $\xi_2^{a(s), b(t-1)}$ with the symmetry of $\begin{array}{|c|} \hline s \\ \hline t-3 \\ \hline \end{array}$ vanish. Both $\xi_1^{a(s-1), b(t)}$ and $\xi_2^{a(s), b(t-1)}$ have traces with the symmetry of $\begin{array}{|c|} \hline s-2 \\ \hline t-1 \\ \hline \end{array}$ and $\begin{array}{|c|} \hline s-1 \\ \hline t-2 \\ \hline \end{array}$, (1.9) and (1.10) imply that these traces are not independent, being proportional to each other.

In contrast to both symmetric and antisymmetric fields there are two gauge parameters, whose Young diagrams can be obtained by cutting off one cell from the

⁹Some work with Young symmetrizers is needed to extract the formulae for particular mixed-symmetry field from the results of [6]. For example, the last term on the *r.h.s.* of (1.5) supplements the second one for the whole expression to have the symmetry of $\phi^{a(s), b(t)}$. In the general case, necessity for Young symmetrizers greatly complicates the issue of extracting given mixed-symmetry field out of [6].

spin Young diagram $\begin{smallmatrix} s \\ t \end{smallmatrix}$ in various ways, i.e., $\begin{smallmatrix} s-1 \\ t \end{smallmatrix}$ and $\begin{smallmatrix} s \\ t-1 \end{smallmatrix}$. However, by the construction the traces of $\xi_1^{a(s-1),b(t)}$ and $\xi_2^{a(s),b(t-1)}$ are not independent. This peculiarity of algebraic conditions will get a simple explanation within the *unfolded* approach. Similarly to p -form fields the gauge symmetry is reducible

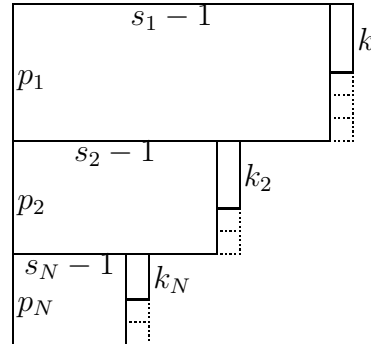
$$\delta\phi^{a(s),b(t)} = 0 \quad \text{provided} \quad \begin{cases} \delta\xi_1^{a(s-1),b(t)} = \partial^b \chi^{a(s-1),b(t-1)} - \frac{1}{s-t} \partial^a \chi^{a(s-2)b,b(t-1)}, \\ \delta\xi_2^{a(s),b(t-1)} = \partial^a \chi^{a(s-1),b(t-1)} \end{cases} \quad (1.11)$$

where the second order gauge parameter $\chi^{a(s-1),b(t-1)}$ is traceless with respect to each pair of indices and has the symmetry of $\begin{smallmatrix} s-1 \\ t-1 \end{smallmatrix}$, i.e., $\chi^{a(s-1),ab(t-2)} \equiv 0$.

A general mixed-symmetry field. In the general case of a spin- $\mathbf{Y} = \mathbf{Y}\{s_1, \dots, s_n\}$ massless field, field $\phi_{\mathbf{Y}}$ is a Lorentz tensor with the symmetry of \mathbf{Y} and in addition to the Young symmetry constraints $\phi^{a(s_1),b(s_2),\dots,d(s_n)}$ satisfies [5]

$$\eta_{cc}\eta_{dd}\phi^{a(s_1),\dots,b(s_i-4)ccdd,\dots,f(s_n)} \equiv 0, \quad i \in [1, n], \quad (1.12)$$

i.e., the second trace with respect to any four indices from the same group of symmetric indices must vanish. Gauge parameters at the r -th level of reducibility have the symmetry of



$$: \sum_{i=1}^{i=N} k_i = \sum_{i=1}^{i=N} p_i - r, \quad (1.13)$$

where it is convenient to combine the rows of equal length into blocks, i.e., $\mathbf{Y} = \mathbf{Y}\{(s_1, p_1), \dots, (s_N, p_N)\}$. Therefore, the first level gauge parameters are obtained by cutting off one cell from the bottom-right of any block. Evidently, there are N gauge parameters at the first level. The trace conditions on the first level gauge parameters are more complicated than (1.12) (cf. (1.6)-(1.10)), see [6, 14] for more details, the general statement is that the traces of gauge parameters are not independent. This fact suggests that all gauge parameters can be incorporated into a single object as they do within the *unfolded* approach.

2 Essential of the Unfolded approach

The *unfolded* approach [42, 43, 44] is a reformulation of field-theoretical systems as

$$dW^{\mathcal{A}} = F^{\mathcal{A}}(W), \quad (2.1)$$

where $W^{\mathcal{A}}$ is a set of differential forms over the space-time, which do not necessary have the same degree, $q_{\mathcal{A}}$ being the degree of $W^{\mathcal{A}}$, d - exterior differential on the space-time manifold and $F^{\mathcal{A}}(W)$ is a degree- $(q_{\mathcal{A}} + 1)$ function that is assumed to be expandable in terms of wedge products only¹⁰, i.e.,

$$F^{\mathcal{A}}(W) = \sum_{n=1}^{\infty} \sum_{q_{\mathcal{B}_1} + \dots + q_{\mathcal{B}_n} = q_{\mathcal{A}} + 1} f^{\mathcal{A}}_{\mathcal{B}_1 \dots \mathcal{B}_n} W^{\mathcal{B}_1} \wedge \dots \wedge W^{\mathcal{B}_n}, \quad (2.2)$$

where $f^{\mathcal{A}}_{\mathcal{B}_1 \dots \mathcal{B}_n}$ are constant coefficients satisfying $f^{\mathcal{A}}_{\mathcal{B}_1 \dots \mathcal{B}_i \mathcal{B}_j \dots \mathcal{B}_n} = (-)^{q_{\mathcal{B}_i} q_{\mathcal{B}_j}} f^{\mathcal{A}}_{\mathcal{B}_1 \dots \mathcal{B}_j \mathcal{B}_i \dots \mathcal{B}_n}$. $F^{\mathcal{A}}(W)$ is also assumed to satisfy an integrability condition, called generalized Jacobi identity,

$$F^{\mathcal{B}} \frac{\vec{\partial}}{\partial W^{\mathcal{B}}} F^{\mathcal{A}} \equiv 0, \quad (2.3)$$

which can be obtained by applying d to (2.1). Any solution of (2.3) is said to define a Free Differential Algebra [45, 46, 47, 48]. In the field-theoretical sense (2.3) corresponds to Bianchi identities.

By virtue of (2.3), equations (2.1) are invariant under gauge transformations

$$\delta W^{\mathcal{A}} = d\xi^{\mathcal{A}} + \xi^{\mathcal{B}} \frac{\vec{\partial}}{\partial W^{\mathcal{B}}} F^{\mathcal{A}}, \quad \text{for } q_{\mathcal{A}} > 0, \quad (2.4)$$

$$\delta W^{\mathcal{A}} = \xi^{\mathcal{B}'} \frac{\vec{\partial}}{\partial W^{\mathcal{B}'}} F^{\mathcal{A}}, \quad \mathcal{B}' : q_{\mathcal{B}'} = 1, \quad \text{for } q_{\mathcal{A}} = 0, \quad (2.5)$$

where $\xi^{\mathcal{A}}$ is a degree- $(q_{\mathcal{A}} - 1)$ form with values in the same space as $W^{\mathcal{A}}$. Due to the fact that each $W^{\mathcal{A}}$ of degree $q_{\mathcal{A}} > 0$ possesses its own gauge parameter $\xi^{\mathcal{A}}$, nonzero degree fields are gauge fields. Provided that $F^{\mathcal{A}}(W)$ is linear in a subset $\omega^i \subset W^{\mathcal{A}}$, $\delta\omega^i = 0$ itself can be treated as unfolded system with respect to ξ^i and, hence, the gauge transformations that does not affect the field ω^i , i.e., the second level gauge transformations, are of the form

$$\delta\xi^i = d\chi^i - \chi^j \frac{\vec{\partial}}{\partial \omega^j} F^i, \quad (2.6)$$

where χ^i is a degree- $(q_i - 2)$ form, and so on for deeper levels gauge symmetries. Therefore, a degree- q_i gauge field ω^i have exactly q_i levels of gauge transformations.

Consequently, equations of motions, gauge symmetries, reducible gauge symmetries and Bianchi identities are the simple consequences of the single algebraic identity (2.3). For example, if we are given some gauge transformations the equations and the Bianchi identities can be easily recovered. In addition, the use of differential forms makes *unfolded* approach manifestly covariant.

The most simple example of an unfolded system is given by one-form connection Ω_{μ}^I of some Lie algebra \mathfrak{g} with structure coefficients f_{JK}^I , satisfying

$$d\Omega^I = -f_{JK}^I \Omega^J \Omega^K. \quad (2.7)$$

¹⁰Further the wedge symbol \wedge will be systematically omitted.

The generalized Jacobi identity reduces to the ordinary Jacobi identity on f_{JK}^I and (2.7) is a zero Yang-Mills strength (flatness) condition on field Ω_μ^I . We take Ω_μ^I to be a connection of the Poincare algebra, i.e., $\Omega^I = \{\varpi^{a,b}, h^a\}$, where $\varpi^{a,b} \equiv \varpi_\mu^{a,b}$ is a Lorentz spin-connection and $h^a \equiv h_\mu^a$ is a background vielbein. Any nontrivial solution (h_μ^a is a nonsingular matrix) of flatness equations

$$dh^a + \varpi^a{}_b h^b = 0, \quad (2.8)$$

$$d\varpi^{a,b} + \varpi^a{}_c \varpi^{c,b} = 0 \quad (2.9)$$

provides us with the basis of a tangent space h_μ^a and with Lorentz spin-connection $\varpi_\mu^{a,b}$, the latter is used to define a Lorentz covariant derivative of differential forms with values in any representation of $\mathfrak{so}(d-1, 1)$, i.e., having some tangent indices,

$$DT^{ab\dots} = dT^{ab\dots} + \varpi^a{}_c T^{cb\dots} + \varpi^b{}_c T^{ac\dots} + \dots \quad (2.10)$$

A simple solution of (2.8-2.9) is given by Cartesian coordinates $h_\mu^a = \delta_\mu^a$, $\varpi_\mu^{a,b} = 0$. It is assumed further that h_μ^a and $\varpi_\mu^{a,b}$ satisfy (2.8-2.9) but the advantage of the *unfolded* approach is that no explicit solution is needed either to write down field-equations or to construct actions, being most effective in (anti)-de Sitter [34, 37, 38, 39, 40].

The set of forms $W^{\mathcal{A}}$ consists of a background connection Ω^I and dynamical fields ω^i , i.e., $W^{\mathcal{A}} = \{\{\varpi^{a,b}, h^a\}, \omega^i\}$. As free fields only are considered, h_μ^a and $\varpi_\mu^{a,b}$ are assumed to be of order zero and ω^i are of the first order, with the equations on ω^i being linear in ω^i . The dynamical fields ω^i are certain forms with values in finite-dimensional irreducible representations of $\mathfrak{so}(d-1, 1)$, i.e., just irreducible Lorentz tensors. The most general linear unfolded equations on ω^i are of the form

$$D\omega^i = - \sum_j \sigma^{ij}(h, \dots, h, \omega^j), \quad (2.11)$$

where the part of the Poincare connection associated with the Lorentz-spin connection $\varpi^{a,b}$ is combined with d into the Lorentz covariant derivative D , which acts diagonally on ω^i ; σ^{ij} are certain operators, which can be polynomial in h^a and satisfy (2.3) since (2.8-2.9) are equivalent to $Dh^a = 0$ and $D^2 = 0$. No wonder that polynomial in vielbein h_μ^a can appear since field equations contain $g_{\mu\nu} g^{\lambda\rho} \phi_{\lambda\rho}$ -like terms, which are polynomial in background metric $g_{\mu\nu}$. Free Differential Algebras are closely connected with Lie algebras, in fact, coefficients of σ^{ij} correspond either to modules or to Chevalley-Eilenberg cocycles of the Poincare algebra [56].

It turns out [55] that the unfolded equations for massless mixed-symmetry fields in Minkowski space are simpler than (2.11) and have the form

$$D\omega^g = -\sigma_-^g(h, \dots, h, \omega^{g+1}), \quad g = 0, 1, \dots, \quad (2.12)$$

where (2.3) reduces to $\sigma_-^g(h, \dots, h, \sigma_-^{g+1}(h, \dots, h, \omega^{g+2})) \equiv 0$ and the subscript $-$ is due to the fact that all fields can be graded by non-negative integer g and σ_- is of grade (-1) . Dynamical field ω^g is a degree- g form that takes values in irreducible representation \mathbf{Y}_g of $\mathfrak{so}(d-1, 1)$, i.e., the tangent tensor has the symmetry of Young

diagram \mathbf{Y}_g and is completely traceless. The rank of tensor \mathbf{Y}_g is a strictly increasing function of g . The operator σ_-^g is uniquely determined by \mathbf{Y}_g and \mathbf{Y}_{g+1} , it contracts each extra index of \mathbf{Y}_{g+1} as compared to \mathbf{Y}_g with h^a and imposes, if needed, appropriate Young symmetrizer¹¹. Therefore, q_g is also completely determined by its value q_0 at $g = 0$, $(q_g - q_{g+1} + 1)$ being equal to the number of vielbeins to be contracted. Young diagrams \mathbf{Y}_g are determined by the spin \mathbf{Y} of the *metric-like* field $\phi_{\mathbf{Y}}(x)$.

If we denote $\sigma_-^g(h, \dots, h, \omega^{g+1})$ simply as $\sigma_-(\omega^{g+1})$, (2.3) is just a nilpotency of σ_- , $\sigma_-^2 = 0$. Technically, the nilpotency of σ_- is due to the fact that when applied twice to any field, say¹² $\omega_{q_g}^{\mathbf{Y}_g}$, it contracts two vielbeins with the same group of symmetric indices, which is identically zero because of $h^a h^b = -h^b h^a$.

It is convenient to define the space of gauge fields $\mathcal{W}_{q_0} = \{\omega_{q_0}^{\mathbf{Y}_0}, \omega_{q_1}^{\mathbf{Y}_1}, \dots\}$, the space of the first level gauge parameters $\mathcal{W}_{q_0-1} = \{\xi_{q_0-1}^{\mathbf{Y}_0}, \xi_{q_1-1}^{\mathbf{Y}_1}, \dots\}$, which contains a finite number of parameters due to the fact that $q_g = 0$ for sufficiently large g . Analogously, we can introduce the spaces \mathcal{W}_{q_0-k} of deeper levels gauge parameters and the space of field strengths $\mathcal{W}_{q_0+1} = \{R_{q_0+1}^{\mathbf{Y}_0}, R_{q_1+1}^{\mathbf{Y}_1}, \dots\}$, the spaces of Bianchi identities $\mathcal{W}_{q_0+k} = \{B_{q_0+k}^{\mathbf{Y}_0}, B_{q_1+k}^{\mathbf{Y}_1}, \dots\}$, $k > 1$. With these definitions the unfolded equations and gauge transformations can be rewritten simply as

$$\begin{aligned} \mathcal{D}\omega_{\mathbf{p}} &= 0, & \omega_{\mathbf{p}} &\in \mathcal{W}_{\mathbf{p}}, \\ \delta\omega_{\mathbf{p}} &= \mathcal{D}\xi_{\mathbf{p}-1}, & \xi_{\mathbf{p}-1} &\in \mathcal{W}_{\mathbf{p}-1}, \\ \delta\xi_{\mathbf{p}-1} &= \mathcal{D}\xi_{\mathbf{p}-2}, & \xi_{\mathbf{p}-2} &\in \mathcal{W}_{\mathbf{p}-2}, \\ &\dots, & & \dots, \\ \delta\xi_{\mathbf{1}} &= \mathcal{D}\xi_0, & \xi_0 &\in \mathcal{W}_0, \end{aligned} \quad (2.13)$$

where $\mathcal{D} = D + \sigma_-$ is a nilpotent operator $\mathcal{D}^2 = 0$. The nilpotency of \mathcal{D} is a simple consequence of (i) nilpotency of D , $(D)^2 = 0$ (2.9); (ii) zero-torsion $Dh^a = 0$ (2.8); (iii) $(\sigma_-)^2 = 0$. Field strength $R_{\mathbf{p}+1} = \mathcal{D}\omega_{\mathbf{p}}$ satisfies Bianchi identity $B_{\mathbf{p}+2} = \mathcal{D}R_{\mathbf{p}+1} \equiv 0$.

Inasmuch as unfolded equations $\mathcal{D}\omega_{\mathbf{p}} = 0$ can be treated [57] as a sort of cocycle condition $d\omega = 0$, by virtue of the Poincare lemma it follows that all fields except for zero-degree forms are pure gauge and, hence, in order for unfolded equations to describe a field-theoretical system with propagating degrees of freedom, zero-forms have to be included.

A spin-two field [58, 42, 43, 37, 53]. To make the above statements more clear let us consider the example of a free massless spin-two field, i.e., the linearized gravitation. The field $\omega_{q_0}^{\mathbf{Y}_0}$ is a well-known dynamical vielbein $e_1^a \equiv e_\mu^a dx^\mu$, the auxiliary field $\omega_{q_1}^{\mathbf{Y}_1}$ is a dynamical Lorentz spin-connection $\omega_1^{ab} \equiv \omega_\mu^{ab} dx^\mu = -\omega_\mu^{ba} dx^\mu$, the first unfolded equation

$$De_1^a + h_c \omega_1^{ac} = 0 \quad (2.14)$$

¹¹For instance, let ω^0 and ω^1 be irreducible Lorentz tensors with the symmetry of \boxplus and \boxtimes , respectively, i.e., $\omega^0 \equiv \omega^{a,b} = -\omega^{b,a}$ and $\omega^1 \equiv \omega^{ab,c} = \omega^{ba,c}$; $\omega^{(ab,c)} \equiv 0$. Then, $D\omega^0 = \sigma_-(\omega^1)$ with $\sigma_-(\omega^1) = h_c \omega^{ac,b} + \frac{1}{2} h_c \omega^{ab,c}$, where the second term (Young symmetrizer) is added in order for the whole expression to have the symmetry of ω^0 .

¹²A degree- q form with values in irreducible representation \mathbf{Y} of $\mathfrak{so}(d-1, 1)$, i.e., the tangent tensor has the symmetry of \mathbf{Y} and is traceless, is denoted as $\omega_{\mathbf{q}}^{\mathbf{Y}}$.

is invariant under gauge transformations

$$\delta e_1^a = D\xi_0^a + h_c \xi_0^{ac}, \quad \delta \omega_1^{ab} = D\xi_0^{ab}, \quad (2.15)$$

where $\xi_0^a \equiv \xi^a$ and $\xi_0^{ab} \equiv \xi^{ab}$ are the gauge parameters associated with e_1^a and ω_1^{ab} , respectively. We have to find $F^A(W)$ in the sector of ω_1^{ab} such that (i) linearized Einstein equations are imposed; (ii) Bianchi identities $h_c D\omega_1^{ac} \equiv 0$ are satisfied. $D\omega_1^{ab}$ (in components $\frac{1}{2}(D_\mu \omega_\nu^{ab} - D_\nu \omega_\mu^{ab})$) is equal to the linearized Riemann tensor $R_{\mu\nu}^{ab}$, where it is useful to introduce its tangent version $R^{ab,cd} = R_{\mu\nu}^{ab} h^{\mu c} h^{\nu d}$. Linearized Einstein equations $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} R_{\lambda\rho} g^{\lambda\rho} = 0$ reduce to $R^{ac,bd} \eta_{cd} = 0$, i.e., the trace of the Riemann tensor is zero but the traceless part of $R^{ab,cd}$, called Weyl tensor $C^{ab,cd}$, need not vanish. If we impose $D\omega_1^{ab} = 0$, then, in addition to the linearized Einstein equations the Weyl tensor is also required to vanish, making the system dynamical empty. A Weyl tensor has the symmetry of \boxplus , i.e., it is antisymmetric in pairs ab , cd and $C^{[ab,c]d} \equiv 0$. Therefore, the correct unfolded equations have the form

$$D\omega_1^{ab} = -h_c h_d C_0^{ab,cd}, \quad (2.16)$$

where a zero-form $C_0^{ab,cd}$ can be recognized as a Weyl tensor, which is the only component of $R^{ab,cd}$ that can be nontrivial on-mass-shell. Bianchi identities $h_b D\omega_1^{ab} = h_b h_c h_d C^{ab,cd} \equiv 0$ are satisfied due to the Young symmetry of a Weyl tensor and the fact that vielbeins h^a anti-commute. Again, $DC^{ab,cd} = 0$ can not be imposed as it constraints the Weyl tensor to be a constant, and so on. The full unfolded system has the form [43]

$$\begin{aligned} De_1^a + h_c \omega_1^{ac} &= 0, & \delta e_1^a &= D\xi_0^a + h_c \xi_0^{ac}, \\ D\omega_1^{ab} + h_c h_d C_0^{ab,cd} &= 0, & \delta \omega_1^{ab} &= D\xi_0^{ab}, \\ DC_0^{aa,bb,c(k)} + h_d C_0^{aa,bb,c(k)d} &= 0, & \delta C_0^{aa,bb,c(k)} &= 0, \quad k = 0, 1, \dots, \end{aligned} \quad (2.17)$$

where a set of zero-forms $C_0^{aa,bb,c(k)}$ that are irreducible Lorentz tensors with the symmetry¹³ of $\boxplus \begin{smallmatrix} k \\ \hline \end{smallmatrix}$ is to be introduced.

The identification with the *metric-like* approach is as follows. The field $e^{a|b} = e_\mu^a h^{\mu b}$ is to be decomposed as $e^{a|b} = \frac{1}{2}\phi^{(ab)} + \frac{1}{2}\psi^{[ab]}$, where $\psi^{[ab]}$ and $\phi^{(ab)}$ are an antisymmetric and a traceful symmetric fields, respectively. In terms of ϕ^{ab} and ψ^{ab} gauge transformations (2.15) read $\delta\phi^{ab} = \partial^a \xi^b + \partial^b \xi^a$, $\delta\psi^{ab} = \partial^a \xi^b - \partial^b \xi^a + 2\xi^{ab}$. By virtue of a pure algebraic gauge symmetry with ξ^{ab} field ψ^{ab} can be gauged away. Therefore, the dynamical field is ϕ^{ab} , which can be recognized as the Fronsdal field (1.1).

Since field equations are of the second order, (2.12) at $g = 0$ expresses the first auxiliary field in terms of the first derivatives of dynamical Labastida field $\phi_{\mathbf{Y}} \in \omega_{\mathbf{q}_0}^{\mathbf{Y}_0}$, (2.12) at $g = 1$ imposes dynamical equations on $\phi_{\mathbf{Y}}$ and expresses the second auxiliary field in terms of the second derivatives of $\phi_{\mathbf{Y}}$, higher equations ($g > 1$) impose no nontrivial constraints on $\phi_{\mathbf{Y}}$. Further, we concentrate on the first two unfolded equations only, since it is sufficient for constructing Lagrangians.

¹³ $C^{aa,bb,c(k)}$ is antisymmetric in pairs aa , bb , symmetric in $c_1 \dots c_k$, traceless with respect to each pair of indices and satisfies $C^{[aa,a]b,c(k)} \equiv 0$, $C^{aa,[bb,b]c(k-1)} \equiv 0$.

A spin- s field [58, 42, 43, 37, 53]. The above can be generalized to a totally-symmetric spin- s field. In accordance with the general construction of [55], $\mathbf{Y}_0 = \boxed{s-1}$, $\mathbf{Y}_1 = \boxed{s-1}$, $\mathbf{Y}_2 = \boxed{s-1}$ and $q_0 = q_1 = q_2 = 1$. Therefore, the dynamical field $\phi^{a(s)}$ is incorporated into a frame-like one-form $e_1^{a(s-1)} \equiv e_\mu^{a(s-1)} dx^\mu$, the first auxiliary field is $\omega_1^{a(s-1),b}$ and the first two unfolded equations together with the gauge transformations have the form

$$\begin{aligned} De_1^{a(s-1)} + h_c \omega_1^{a(s-1),c} &= 0, & \delta e_1^{a(s-1)} &= D\xi_0^{a(s-1)} + h_c \xi_0^{a(s-1),c}, \\ D\omega_1^{a(s-1),b} + h_c \omega_1^{a(s-1),bc} &= 0, & \delta \omega_1^{a(s-1),b} &= D\xi_0^{a(s-1),b} + h_c \xi_0^{a(s-1),bc}. \end{aligned} \quad (2.18)$$

Similarly to the spin-two case $e^{a(s-1)|b} = e_\mu^{a(s-1)} h^{\mu b}$ decomposes into two fields $e^{a(s-1)|b} = \frac{1}{s} \phi^{a(s-1)b} + \frac{1}{s} \psi^{a(s-1),b}$ with the symmetry of \boxed{s} and $\boxed{s-1}$, respectively. Note that $e^{a(s-1)|b}$ is traceless with respect to a_1, \dots, a_{s-1} and, hence, the second trace of $\phi^{a(s)}$ vanishes. In terms of ϕ and ψ the gauge transformations have the form $\delta \phi^{a(s)} = \partial^a \xi^{a(s-1)}$ and $\delta \psi^{a(s-1),b} = (s-1) \partial^b \xi^{a(s-1)} - \partial^a \xi^{a(s-2)b} + s \xi^{a(s-1),b}$. Therefore, the traceless part of $\psi^{a(s-1),b}$ can be gauged away¹⁴ by virtue of $\xi^{a(s-1),b}$. With world indices converted to the tangent ones according to $\omega^{a(s-1),b|c} \equiv \omega_\mu^{a(s-1),b} h^{c\mu}$, $\omega^{a(s-1),bb|c} \equiv \omega_\mu^{a(s-1),bb} h^{c\mu}$, the first two unfolded equations (2.18) are rewritten as

$$\partial^c e^{a(s-1)|d} - \partial^d e^{a(s-1)|c} = \omega^{a(s-1),c|d} - \omega^{a(s-1),d|c}, \quad (2.19)$$

$$\partial^c \omega^{a(s-1),b|d} - \partial^d \omega^{a(s-1),b|c} = \omega^{a(s-1),bc|d} - \omega^{a(s-1),bd|c}. \quad (2.20)$$

Symmetrizing c with $a_1 \dots a_{s-1}$ in (2.19), (2.20) and, then, contracting (2.20) with η_{bd} results in

$$\omega^{a(s-1),d|a} = \partial^d e^{a(s-1)|a} - \partial^a e^{a(s-1)|d}, \quad (2.21)$$

$$\omega^{a(s-1),m|}_m = \partial_m e^{a(s-2)m|a} - \partial^a e^{a(s-2)m|}_m, \quad (2.22)$$

$$\partial_m \omega^{a(s-1),m|a} - \partial^a \omega^{a(s-1),m|}_m = 0, \quad (2.23)$$

where (2.22) is a contraction of (2.21) with η_{da} . Note that $\omega_1^{a(s-1),bc}$ does not contribute to the dynamical equations of motion since it is auxiliary. Substituting (2.21), (2.22) into (2.23) and, then, making use of $\phi^{a(s)} = e^{a(s-1)|a}$ and $\phi^m_{m a(s-2)} = 2e^{a(s-2)m|}_m$ gives the Fronsdal equations (1.1) for $\phi^{a(s)}$.

A spin- $\mathbf{Y}\{s, t\}$ field. According to the general recipe of [55] $\mathbf{Y}_0 = \boxed{s-1}$, $\mathbf{Y}_1 = \boxed{s-1}$, $\mathbf{Y}_2 = \boxed{s-1}$ and $q_0 = q_1 = q_2 = 2$. The Labastida field $\phi^{a(s),b(t)}$ is to be incorporated into the frame-like field $e_2^{a(s-1),b(t-1)}$ and an analog of Lorentz spin-connection is given by $\omega_2^{a(s-1),b(t-1),c}$. The first two unfolded equations

$$R_3^{a(s-1),b(t-1)} \equiv De_2^{a(s-1),b(t-1)} + h_d \omega_2^{a(s-1),b(t-1),d} = 0, \quad (2.24)$$

$$R_3^{a(s-1),b(t-1),c} \equiv D\omega_2^{a(s-1),b(t-1),c} + h_d \omega_2^{a(s-1),b(t-1),cd} = 0, \quad (2.25)$$

¹⁴Note that the trace of the Fronsdal field $\phi^{a(s)}$ is also contained in $\psi^{a(s-1),b}$, namely, $2\phi^{a(s-2)m|}_m = (s-2)\psi^{a(s-2)m|}_m$. Nevertheless, the dynamical components of $e_1^{a(s-1)}$ are equivalent to the Fronsdal field $\phi^{a(s)}$.

are obviously invariant under the first level gauge transformations

$$\delta e^{a(s-1),b(t-1)} = D\xi_1^{a(s-1),b(t-1)} + h_d \xi_1^{a(s-1),b(t-1),d}, \quad (2.26)$$

$$\delta \omega^{a(s-1),b(t-1),c} = D\xi_1^{a(s-1),b(t-1),c} + h_d \xi_1^{a(s-1),b(t-1),cd}, \quad (2.27)$$

$$\delta \omega^{a(s-1),b(t-1),cc} = D\xi_1^{a(s-1),b(t-1),cc} + h_d \xi_1^{a(s-1),b(t-1),ccd}, \quad (2.28)$$

where the last term in (2.28) is related to higher-grade fields and is not needed in what follows, the only point being that it has $\sigma_-(\dots)$ form.

In its turn the gauge fields are invariant under the second level gauge transformations

$$\delta \xi_1^{a(s-1),b(t-1)} = D\chi_0^{a(s-1),b(t-1)} + h_d \chi_0^{a(s-1),b(t-1),d}, \quad (2.29)$$

$$\delta \xi_1^{a(s-1),b(t-1),c} = D\chi_0^{a(s-1),b(t-1),c} + h_d \chi_0^{a(s-1),b(t-1),cd}, \quad (2.30)$$

$$\delta \xi_1^{a(s-1),b(t-1),cc} = D\chi_0^{a(s-1),b(t-1),cc} + h_d \chi_0^{a(s-1),b(t-1),ccd}. \quad (2.31)$$

Let us draw the reader's attention to the conformity of equations with gauge transformations. Note that gauge fields/parameters take values in irreducible representations of the Lorentz algebra, i.e., the tangent tensors have definite Young symmetry and are completely traceless. $e^{a(s-1),b(t-1)|cd} = e_{\mu\nu}^{a(s-1),b(t-1)} h^{\mu c} h^{\nu d}$ can easily be decomposed into irreducible tensors of $\mathfrak{so}(d-1, 1)$ as

$$\begin{aligned} \begin{array}{|c|c|} \hline s- \\ \hline t-1 \\ \hline \end{array} \otimes \begin{array}{|c|} \hline \\ \hline \end{array} &= \left[\begin{array}{|c|c|} \hline s- \\ \hline t \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline s- \\ \hline t-1 \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline s- \\ \hline t-2 \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline s-2 \\ \hline t \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline s-2 \\ \hline t-2 \\ \hline \end{array} \right] \oplus \\ &\oplus \left[\begin{array}{|c|c|} \hline s \\ \hline t-1 \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline s-1 \\ \hline t \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline s- \\ \hline t-1 \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline s-2 \\ \hline t-1 \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline s-1 \\ \hline t-2 \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline s-1 \\ \hline t-1 \\ \hline \end{array} \right], \quad (2.32) \end{aligned}$$

where all Young diagrams correspond to traceless tensors. The irreducible tensors in the first brackets of (2.32) are the components of the Labastida field $\phi^{a(s),b(t)} = e^{a(s-1),b(t-1)|ab}$. Indeed, by virtue of the second level gauge parameter $\chi_0^{a(s-1),b(t-1),cd}$

with the symmetry of $\begin{array}{|c|c|} \hline s- \\ \hline t-1 \\ \hline \end{array}$ the respective component of the first order gauge parameter $\xi^{a(s-1),b(t-1),c|d} = \xi_{\mu}^{a(s-1),b(t-1),c} h^{\mu d}$ can be gauged away, with the rest of irreducible components of $\xi^{a(s-1),b(t-1),c|d}$ exactly matching the terms in the second brackets of (2.32). Therefore, it is $\phi^{a(s),b(t)} = e^{a(s-1),b(t-1)|ab}$ that can not be gauged away by an algebraic gauge symmetry, thus being a dynamical field. Note that there are two irreducible components with the symmetry of $\begin{array}{|c|c|} \hline s- \\ \hline t-1 \\ \hline \end{array}$, with one belonging to $\phi^{a(s),b(t)}$ and the other one being pure gauge.

The Labastida double-tracelessness (1.3) is a consequence of the irreducibility of tangent tensors and of the lack of any trace conditions with respect to world and tangent indices. The Labastida gauge parameters $\xi_1^{a(s-1),b(t)}$ and $\xi_2^{a(s),b(t-1)}$ can be identified as

$$\xi_1^{a(s-1),b(t)} = \xi^{a(s-1),b(t-1)|b} + \frac{1}{s-t} \xi^{a(s-2)b,b(t-1)|a}, \quad (2.33)$$

$$\xi_2^{a(s),b(t-1)} = \xi^{a(s-1),b(t-1)|a}, \quad (2.34)$$

then, nontrivial trace constraints (1.6-1.10) are the consequence of this definition. To obtain, if needed at all, the Labastida equations (1.4) from the unfolded ones (2.24-2.25) is a simple 'symmetrizing and taking traces' problem.

From unfolded to metric-like for a spin-Y $\{s, t\}$ field. We work in Cartesian coordinates, i.e., $h_\mu^a = \delta_\mu^a$, $\varpi_\mu^{a,b} = 0$ and, hence, there is no distinction between world and tangent indices. With world indices converted to the tangent ones, the first two unfolded equations (2.24)-(2.25) read

$$\partial^{[c} e^{a(s-1), b(t-1)]|cc} = -\omega^{a(s-1), b(t-1), [c|cc]}, \quad (2.35)$$

$$\partial^{[c} \omega^{a(s-1), b(t-1), d|cc} = -\omega^{a(s-1), b(t-1), d[c|cc]}, \quad (2.36)$$

Symmetrizing c_1, c_2 with $a_1 \dots a_{s-1}$ and $b_1 \dots b_{t-1}$ in (2.35-2.36), and taking trace with respect to d and c in (2.36) results in

$$\partial^a e^{a(s-1), b(t-1)|bc} + \partial^b e^{a(s-1), b(t-1)|ca} + \partial^c e^{a(s-1), b(t-1)|ab} = -\omega^{a(s-1), b(t-1), c|ab}, \quad (2.37)$$

$$\partial^a \omega^{a(s-1), b(t-1), m|b}_m + \partial^b \omega^{a(s-1), b(t-1), m|}_m^a + \partial_m \omega^{a(s-1), b(t-1), m|ab} = 0. \quad (2.38)$$

Note that field $\omega_2^{a(s-1), b(t-1), cc}$ is auxiliary and does not contribute to the dynamical equations, being excluded by the projector onto the Labastida equations. The third term in (2.38) can be directly expressed from (2.37); to express the first two terms of (2.38) we take the traces in (2.37) with respect to a_s, c and b_t, c

$$\partial^a e^{a(s-2)m, b(t-1)|b}_m + \partial^b e^{a(s-2)m, b(t-1)|}_m^a + \partial_m e^{a(s-2)m, b(t-1)|ab} = -\omega^{a(s-1), b(t-1), m|}_m^b, \quad (2.39)$$

$$\partial^a e^{a(s-1), b(t-2)m|b}_m + \partial^b e^{a(s-1), b(t-2)m|}_m^a + \partial_m e^{a(s-1), b(t-2)m|ab} = -\omega^{a(s-1), b(t-1), m|a}_m. \quad (2.40)$$

The Labastida field $\phi^{a(s), b(t)}$ is to be identified as $\phi^{a(s), b(t)} = e^{a(s-1), b(t-1)|ab}$, its various traces are given by

$$\begin{aligned} \phi^{a(s-2)m, b(t)}_m &= 2e^{a(s-2)m, b(t-1)|}_m^b, & \phi^{a(s), b(t-2)m}_m &= 2e^{a(s-1), b(t-2)m|a}_m, \\ \phi^{a(s-1)m, b(t-1)}_m &= e^{a(s-2)m, b(t-1)|a}_m + e^{a(s-1), b(t-2)m|}_m^b. \end{aligned} \quad (2.41)$$

Substituting (2.39), (2.40), (2.37) into (2.38) and, then, using (2.41) results in the Labastida equations (1.4).

For the further comparison with the Lagrangian equations let us note that the dynamical equations have been obtained by making use of $R^{a(s-1), b(t-1)|abc}$ and $R^{a(s-1), b(t-1), m|ab}_m$ components of field strengths (2.24) and (2.24), instead of the whole field strengths. The auxiliary field $\omega_2^{a(s-1), b(t-1), cc}$ does not contribute to $R^{a(s-1), b(t-1), m|ab}_m$ by the construction.

It is significant that the above computations can be directly generalized to an arbitrary-spin mixed-symmetry field, resulting in the Labastida equations [6].

A spin- \mathbf{Y} field. The main statement of [55] is that a spin- \mathbf{Y}^{15} massless field can be uniquely described within the *unfolded* approach. \mathbf{Y}_0 is obtained by cutting off the first column of \mathbf{Y} , i.e., $\mathbf{Y}_0 = \mathbf{Y}\{(s_1 - 1, p_1), \dots, (s_N - 1, p_N)\} = \mathbf{Y}\{h_2, h_3, \dots, h_{s_1}\}$, $\mathbf{Y}_1 = \mathbf{Y}\{h_1 + 1, h_3, \dots, h_{s_1}\}$, $\mathbf{Y}_2 = \mathbf{Y}\{h_1 + 1, h_2 + 1, h_4, \dots, h_{s_1}\}$. $q_0 = p = h_1$, $q_1 = h_2$, $q_2 = h_3$. For instance, when $s_N > 2$

$$\mathbf{Y}_0 = \begin{array}{|c|} \hline s_1-1 \\ \hline p_1 \\ \hline s_2-1 \\ \hline p_2 \\ \hline s_N-1 \\ \hline p_N \\ \hline \end{array} \quad \mathbf{Y}_1 = \begin{array}{|c|} \hline s_1-1 \\ \hline p_1 \\ \hline s_2-1 \\ \hline p_2 \\ \hline s_N-1 \\ \hline p_N \\ \hline \end{array} \quad \mathbf{Y}_2 = \begin{array}{|c|} \hline s_1-1 \\ \hline p_1 \\ \hline s_2-1 \\ \hline p_2 \\ \hline s_N-1 \\ \hline p_N \\ \hline \end{array} \quad (2.42)$$

The sketch of the proof

1. In order for gauge transformations to have p levels, the Labastida field $\phi_{\mathbf{Y}}$ has to be incorporated into certain degree- p form $e_{\mathbf{p}}^{\mathbf{Y}_0}$. The gauge parameter at the p -th level is a degree-zero form $\xi_{\mathbf{0}}^{\mathbf{Y}_0}$ and, hence, \mathbf{Y}_0 has the symmetry of the single gauge parameter at the level- p (cf. (1.13)). Converting all world form indices to the tangent ones

$$e^{a_1(s_1-1), \dots, a_p(s_p-1) | [d_1 \dots d_p]} = e^{a_1(s_1-1), \dots, a_p(s_p-1)} h^{\mu_1 d_1} \dots h^{\mu_p d_p}, \quad (2.43)$$

the field $\phi_{\mathbf{Y}}(x)$ is to be identified with $e^{a_1(s_1-1), \dots, a_p(s_p-1) | a_1 \dots a_p}$. The Labastida double-tracelessness condition is a simple consequence of the tracelessness of the tangent tensor. All level- k gauge parameters are contained in a single object $\xi_{\mathbf{p}-\mathbf{k}}^{\mathbf{Y}_0}$, explaining the fact that the traces of gauge parameters are not independent within the *metric-like* approach.

2. The gauge symmetry was made manifest by the price of introducing redundant components that are given by various components of $e^{a_1(s_1-1), \dots, a_p(s_p-1) | [d_1 \dots d_p]}$ that do not belong to $\phi_{\mathbf{Y}}(x)$, gauge parameters may also have redundant components. In order to make redundant fields non-dynamical, an algebraic (*Stueckelberg*) symmetry is introduced. Fortunately, all redundant components can be compensated by an algebraic gauge symmetry with a single $\xi_{\mathbf{q}_1-1}^{\mathbf{Y}_1}$

$$\delta e_{\mathbf{p}}^{\mathbf{Y}_0} = D \xi_{\mathbf{p}-1}^{\mathbf{Y}_0} + \sigma_{-}(\xi_{\mathbf{q}_1-1}^{\mathbf{Y}_1}), \quad (2.44)$$

analogously for gauge parameters at the level- k

$$\delta \xi_{\mathbf{p}-\mathbf{k}}^{\mathbf{Y}_0} = D \xi_{\mathbf{p}-\mathbf{k}-1}^{\mathbf{Y}_0} + \sigma_{-}(\xi_{\mathbf{q}_1-\mathbf{k}-1}^{\mathbf{Y}_1}). \quad (2.45)$$

3. Since gauge parameter $\xi_{\mathbf{q}_1-1}^{\mathbf{Y}_1}$ is associated with the gauge field $\omega_{\mathbf{q}_1}^{\mathbf{Y}_1}$, the first unfolded equation is

$$D e_{\mathbf{p}}^{\mathbf{Y}_0} + \sigma_{-}(\omega_{\mathbf{q}_1}^{\mathbf{Y}_1}) = 0. \quad (2.46)$$

¹⁵ $\mathbf{Y} = \mathbf{Y}\{(s_1, p_1), \dots, (s_N, p_N)\}$ is taken in block's notation or $\mathbf{Y} = \{h_1, h_2, \dots, h_{s_1}\}$, h_i being the height of the i -th column. Let $p = p_1 + \dots + p_N$ be the height of \mathbf{Y} . It is obvious that $h_1 = p$, $h_2 = p$ if $s_N > 1$ and $h_2 = p - p_N$, otherwise.

4. The Bianchi identities $\sigma_-(D\omega_{\mathbf{q}1}^{\mathbf{Y}_1}) \equiv 0$ can be solved [55] as

$$D\omega_{\mathbf{q}1}^{\mathbf{Y}_1} + \sigma_-(\omega_{\mathbf{q}2}^{\mathbf{Y}_2}) = 0. \quad (2.47)$$

5. Equation (2.47) implies new Bianchi identities and so on, resulting in the full unfolded system [55].

6. Despite the unambiguity of unfolding, the facts that (i) correct second order equations are indeed imposed on the dynamical field $\phi_{\mathbf{Y}} \in e_{\mathbf{p}}^{\mathbf{Y}_0}$; (ii) the rest of the unfolded equations imposes no additional differential constraints on $\phi_{\mathbf{Y}}$; (iii) there are no other dynamical fields in the system; (iv) equations imposed indeed describe the correct number of physical degrees of freedom¹⁶; have to be checked. The σ_- cohomology technique [34, 59] turned out to be very effective, solving all four problems at once [55].

3 Local Actions

To begin with, let us note that the Maxwell action for a spin-one massless field $S = -\frac{1}{4} \int d^d x F_{\mu\nu} F^{\mu\nu}$, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and A_μ is a potential, can be rewritten in the first order form

$$S = \int d^d x (\partial_\mu A_\nu - \partial_\nu A_\mu + C_{\mu\nu}) C^{\mu\nu}, \quad (3.1)$$

where $C_{\mu\nu}$ is a rank-two antisymmetric auxiliary field. The equations for $C_{\mu\nu}$ are algebraic with respect to $C_{\mu\nu}$ and can be easily solved as $C_{\mu\nu} = -\frac{1}{2} F_{\mu\nu}$. The key moment is that (3.1) admits a reformulation in terms of differential forms, i.e., frame-like fields to be embedded into unfolded systems

$$S = \int \left(dA_1 + \frac{1}{2} h_b h_c C^{bc} \right) C^{aa} \epsilon_{aav(d-2)} h^v \dots h^v, \quad (3.2)$$

where $A_1 \equiv A_\mu$ is a Maxwell gauge potential one-form and $C^{ab} = -C^{ba}$ is a degree-zero form, which is antisymmetric in tangent indices and is a tangent version of $C_{\mu\nu}$. $\epsilon_{a_1 \dots a_d}$ is a totally antisymmetric tensor, the Levi-Civita symbol. Use is made of

$$\underbrace{\epsilon_{a_1 \dots a_d} \Xi_{\mathbf{k}}^{a_1 \dots a_k} h^{a_{k+1}} \dots h^{a_d}}_{\text{degree-}d \text{ volume form}} = \left[\begin{array}{c} \text{Cartesian} \\ \text{coordinates} \end{array} \right] = \Xi_{\mu_1 \dots \mu_k}^{a_1 \dots a_k} \delta_{a[k]}^{\mu[k]}, \quad (3.3)$$

where $\Xi_{\mathbf{k}}^{a_1 \dots a_k}$ is a k -from with k antisymmetric tangent indices and $\delta_{a[k]}^{\mu[k]} = \delta_{a_1}^{\mu_1} \dots \delta_{a_k}^{\mu_k}$. A frame-like action for a totally-symmetric spin- $(s > 1)$ field was constructed in [30]

$$S = \int (de^{ua(s-2)} + \frac{1}{2} h_c \omega^{ua(s-2),c}) \omega_{a(s-2)}{}^{u,u} \epsilon_{uuuv(d-3)} h^v \dots h^v, \quad (3.4)$$

¹⁶This problem was not solved by Labastida in general, but was solved in [14]. As it has been already noted the spin- $\mathbf{Y}\{s, t\}$ example can be easily extended to the general case in order to derive the Labastida equations from the unfolded ones for arbitrary-spin field. This provides us with another consistency check of the Labastida work.

which operates with the first two fields, the dynamical one and the first auxiliary one, of unfolded system (2.18) and correctly reproduces the dynamical equations. Action (3.4) is equivalent to the Fronsdal one after solving the algebraic equations for auxiliary field $\omega_\mu^{a(s-1),b}$ and expressing the terms with $e_\mu^{a(s-1)}$ via $\phi_{\mu_1 \dots \mu_s}$.

It is the action (3.4) that will be generalized to the action for an arbitrary mixed-symmetry field. Very instructive was the observation of [35, 36], made on the basis of the simplest mixed-symmetry fields, that the indices of fields can be formally split into world and tangent ones.

Since dynamical equations are of the second order, it is sufficient to make use of gauge fields/field strengths at grade zero and one¹⁷

$$R_{\mathbf{p}+1}^0 = De_{\mathbf{p}}^{\mathbf{Y}_0} + \sigma_-(\omega_{\mathbf{q}}^{\mathbf{Y}_1}), \quad R_{\mathbf{p}+1}^0 \in \mathcal{W}_{\mathbf{p}+1}^{g=0}, \quad e_{\mathbf{p}}^{\mathbf{Y}_0} \in \mathcal{W}_{\mathbf{p}}^{g=0} \quad (3.5)$$

$$R_{\mathbf{q}+1}^1 = D\omega_{\mathbf{q}}^{\mathbf{Y}_1} + \sigma_-(\omega_{\mathbf{r}}^{\mathbf{Y}_2}), \quad R_{\mathbf{q}+1}^1 \in \mathcal{W}_{\mathbf{p}+1}^{g=1}, \quad \omega_{\mathbf{q}}^{\mathbf{Y}_1} \in \mathcal{W}_{\mathbf{p}}^{g=1}, \quad \omega_{\mathbf{r}}^{\mathbf{Y}_2} \in \mathcal{W}_{\mathbf{p}}^{g=2}. \quad (3.6)$$

To construct an action, a degree- d volume form that is bilinear in fields has to be found. It is convenient to introduce for any $k = 0 \dots d$ a degree- $(d-k)$ form with k antisymmetric indices $E_{u[k]} \equiv \epsilon_{u[k]b_1 \dots b_{d-k}} h^{b_1} \dots h^{b_{d-k}}$, built with background vielbeins, which satisfies¹⁸

$$h^c E_{u_1 \dots u_k} = \frac{1}{d-k+1} \sum_{i=1}^{i=k} (-)^{i+k} \delta_{u_i}^c E_{u_1 \dots \hat{u}_i \dots u_k}. \quad (3.7)$$

For a degree- $(p'+1)$ form $\Phi_{\mathbf{p}'+1}^{\mathbf{Y}_0}$ with tangent indices $a(s_1-1), b(s_2-1), \dots, c(s_r-1), d, \dots, e$ and a degree- q' form $\Psi_{\mathbf{q}'}^{\mathbf{Y}_1}$ with tangent indices $a(s_1-1), b(s_2-1), \dots, c(s_r-1), d, e, \dots, f$ provided that $p+q = p' + q'$ let us define a scalar product

$$\langle \Phi_{\mathbf{p}'+1}^{\mathbf{Y}_0} | \Psi_{\mathbf{q}'}^{\mathbf{Y}_1} \rangle = \int \Phi_{\mathbf{p}'+1}^{ua(s_1-2), \dots, uc(s_r-2), \overbrace{u, \dots, u}^{q-r}} \Psi_{\mathbf{q}'}^{u_{a(s_1-2)}, \dots, u_{c(s_r-2)}, \overbrace{u, u, \dots, u}^{p+1-r}} E_{u[p+q+1]}. \quad (3.8)$$

The first index from each of the groups is contracted with $E_{u \dots u}$. Since \mathbf{Y}_0 and \mathbf{Y}_1 coincides modulo first column, the rest of tangent indices of $\Phi_{\mathbf{p}'+1}^{\mathbf{Y}_0}$ are contracted with the corresponding tangent indices of $\Psi_{\mathbf{q}'}^{\mathbf{Y}_1}$.

Taking into account that the sum of the heights of the first columns of \mathbf{Y}_0 and \mathbf{Y}_1 is equal to the total degree $(p+q+1)$ of $\Phi_{\mathbf{p}'+1} \Psi_{\mathbf{q}'}$ and the tangent tensors have definite Young symmetry and are completely traceless, it is easy to see that the scalar product possesses two important properties

$$\langle \sigma_-(\Theta_{\mathbf{q}}^{\mathbf{Y}_1}) | \Psi_{\mathbf{q}}^{\mathbf{Y}_1} \rangle = \langle \sigma_-(\Psi_{\mathbf{q}}^{\mathbf{Y}_1}) | \Theta_{\mathbf{q}}^{\mathbf{Y}_1} \rangle, \quad \text{for any } \Theta_{\mathbf{q}}^{\mathbf{Y}_1}, \Psi_{\mathbf{q}}^{\mathbf{Y}_1} \in \mathcal{W}_{\mathbf{p}}^{g=1}, \quad (3.9)$$

$$\langle \Phi_{\mathbf{p}'}^{\mathbf{Y}_0} | \sigma_-(\Upsilon_{\mathbf{r}'}^{\mathbf{Y}_2}) \rangle = 0, \quad \text{for any } \begin{matrix} \Phi_{\mathbf{p}'}^{\mathbf{Y}_0} \in \mathcal{W}_{\mathbf{p}'}^{g=0} \\ \Upsilon_{\mathbf{r}'}^{\mathbf{Y}_2} \in \mathcal{W}_{2\mathbf{p}-\mathbf{p}'}^{g=2} \end{matrix}, \quad p' + r' = p + r. \quad (3.10)$$

¹⁷Recall that for a spin- \mathbf{Y} field, where $\mathbf{Y} = \mathbf{Y}\{(s_1, p_1), \dots, (s_N, p_N)\} = \mathbf{Y}\{h_1, h_2, \dots, h_{s_1}\}$, $p = p_1 + p_2 + \dots + p_N$ is the height of the first column of \mathbf{Y} ; We abbreviate q_1 and q_2 as q and r , q_0 being equal to p by definition. Note that $q = p = h_2$ unless $s_N = 1$ and $q = p - p_N = h_2$ otherwise. Then, $\mathbf{Y} = \mathbf{Y}\{p, q, r, h_4, \dots, h_{s_1}\}$, $\mathbf{Y}_0 = \mathbf{Y}\{q, r, h_4, \dots, h_{s_1}\}$, $\mathbf{Y}_1 = \mathbf{Y}\{p+1, r, h_4, \dots, h_{s_1}\}$, $\mathbf{Y}_2 = \mathbf{Y}\{p+1, q+1, h_4, \dots, h_{s_1}\}$.

¹⁸(3.7) can be derived from the identity $\epsilon_{[u_1 \dots u_k b_2 \dots b_{d-k+1}] b_1} \delta_{b_1}^c h^{b_1} h^{b_2} \dots h^{b_{d-k}} \equiv 0$.

The proof of (3.9). Let $\Theta_q^{Y_1}$ and $\Psi_q^{Y_1}$ be elements of $\mathcal{W}_p^{g=1}$. Assume that $q = p$. Applying (3.7) one obtains

$$\begin{aligned}
& h_m \Theta_q^{ua(s_1-2), \dots, uc(s_r-2), \overbrace{u, \dots, u}^{q-r}, m} \Psi_q^u_{a(s_1-2) \dots c(s_r-2), \overbrace{u, \dots, u}^{p+1-r}} E_{u[p+1+q]} \sim \\
& \sim \sum_{i=1}^{i=p+1} (-)^i \Theta_q^{ua(s_1-2), \dots, ub(s_i-2), \dots, uc(s_r-2), \overbrace{u, \dots, u}^{p-r}, m} \Psi_q^u_{a(s_1-2) \dots mb(s_i-2) \dots c(s_r-2), \overbrace{u, \dots, u}^{p+1-r}} E_{u[2q]} \sim \\
& \sim \sum_{i=1}^{i=p+1} (-)^i \Theta_q^{ua(s_1-2), \dots, b(s_i-1), \dots, uc(s_r-2), \overbrace{u, \dots, u}^{p+1-r}} \Psi_q^u_{a(s_1-2) \dots b(s_i-1) \dots c(s_r-2), \overbrace{u, \dots, u}^{p+1-r}} E_{u[2q]},
\end{aligned}$$

where the last expression is obviously symmetric with respect to $\Theta_q^{Y_1}$ and $\Psi_q^{Y_1}$, and use is made of the following simple consequence of the Young symmetry properties

$$C^{a(k_1), \dots, b(k_i-1)u, \dots, bc(k_j-1), \dots} = -C^{a(k_1), \dots, b(k_i), \dots, uc(k_j-1), \dots} \quad (3.11)$$

For the case $p > q$, $\sigma_-(\Theta_q^{Y_1})$ involves $(p+1-q)$ vielbeins

$$h_{m_1} \dots h_{m_{p+1-q}} \Theta_q^{ua(s_1-2), \dots, uc(s_r-2), \overbrace{u, \dots, u}^{q-r}, m_1, \dots, m_{p+1-q}}. \quad (3.12)$$

by applying (3.7) m_1, \dots, m_{p+1-q} turn out to be contracted with certain u, \dots, u from the i_1, \dots, i_{p+1-q} groups of $\Psi_p^{Y_1}$, the sum over different rearrangements is implied. If i_k is in the range $1, \dots, r$ then m_{i_k} appears to be symmetrized with $s_{i_k} - 2$ indices of the i_k group and, hence, by virtue of (3.11) m_{i_k} can be exchanged with u_{i_k} , the resulting expression being symmetric in Θ and Ψ . If i_k is in the range $r+1, \dots, p+1$ m_{i_k} can be replaced with u_{i_k} directly since Θ is explicitly antisymmetric in the indices from the groups $r+1, \dots, p+1$ (each group consists only of one index), and hence these terms can also be cast into the form that is explicitly symmetric with respect to Θ and Ψ . ■

The proof of (3.10). Let $\Phi_{p'}^{Y_0}$ and $\Upsilon_{r'}^{Y_2}$ be elements of $\mathcal{W}_{p'}^{g=0}$ and $\mathcal{W}_{2p-p'}^{g=2}$, $p+r = p' + r'$. Assume that $q = r$. Applying (3.7) one obtains

$$\begin{aligned}
& \Phi_{p'}^{ua(s_1-2), \dots, uc(s_r-2)} h_m \Upsilon_{r'}^u_{a(s_1-2) \dots c(s_r-2), \overbrace{u, m, u, \dots, u}^{p-q}} E_{u[p+q+1]} \sim \\
& \sim \sum_{i=1}^{i=r} (-)^i \Phi_{p'}^{ua(s_1-2), \dots, mb(s_i-2), \dots, uc(s_r-2)} \Upsilon_{r'}^u_{a(s_1-2) \dots b(s_i-2) \dots c(s_r-2), \overbrace{u, m, u, \dots, u}^{p-q}} E_{u[p+q]} \sim \\
& \sim \sum_{i=1}^{i=r} (-)^i \Phi_{p'}^{ua(s_1-2), \dots, b(s_i-1), \dots, uc(s_r-2)} \Upsilon_{r'}^u_{a(s_1-2) \dots b(s_i-1) \dots c(s_r-2), \overbrace{u, u, u, \dots, u}^{p-q}} E_{u[p+q]} = 0
\end{aligned}$$

to obtain the last expression, which is identically zero since two antisymmetrized indices u appear in the same group of symmetric indices, use was made of (3.11). The extension on $q > r$ is similar to the proof of (3.9): among $h_{m_1} \dots h_{m_{q-r+1}}$ of

$$\sigma_-(\Upsilon_{r'}^{Y_2}) \longleftrightarrow h_{m_1} \dots h_{m_{q-r+1}} \Upsilon_{r'}^{ua(s_1-2), \dots, uc(s_r-2), \overbrace{u, m, \dots, u, m, u, \dots, u}^{q+1-r}, \overbrace{u, \dots, u}^{p-q}} \quad (3.13)$$

at least one of m will be contracted with some of the first r groups of indices of Φ and hence by virtue of (3.11) it can be exchanged with u , resulting in zero. ■

In order to derive the most general form of the action we use the three crucial observations: (a) (3.8) is the only way to build a degree- d volume form of the elements of $\mathcal{W}_{\mathbf{p}+1}^{g=0}$ and $\mathcal{W}_{\mathbf{p}}^{g=1}$; (b) there are two relevant elements of $\mathcal{W}_{\mathbf{p}+1}^{g=0}$, i.e., $De_{\mathbf{p}}^{\mathbf{Y}_0}$ and $\sigma_-(\omega_{\mathbf{q}}^{\mathbf{Y}_1})$; (c) there is only one relevant element of $\mathcal{W}_{\mathbf{p}}^{g=1}$, i.e., $\omega_{\mathbf{q}}^{\mathbf{Y}_1}$; the most general action is proved to be of the form

$$S = \langle De_{\mathbf{p}}^{\mathbf{Y}_0} + (1 + \alpha)\sigma_-(\omega_{\mathbf{q}}^{\mathbf{Y}_1}) | \omega_{\mathbf{q}}^{\mathbf{Y}_1} \rangle, \quad (3.14)$$

with α being a free coefficient.

Lagrangian equations are to be of the form

$$\frac{\delta S}{\delta \omega_{\mathbf{q}}^{\mathbf{Y}_1}} = \pi_1[R_{\mathbf{p}+1}^0] = 0, \quad (3.15)$$

$$\frac{\delta S}{\delta e_{\mathbf{p}}^{\mathbf{Y}_0}} = \pi_0[R_{\mathbf{q}+1}^1] = \pi_0[D\omega_{\mathbf{q}}^{\mathbf{Y}_1}] = 0, \quad (3.16)$$

where π_1 is a projector into $\omega_{\mathbf{q}}^{\mathbf{Y}_1}$ and π_0 is a projector into $e_{\mathbf{p}}^{\mathbf{Y}_0}$, i.e., π_0 and π_1 are the projectors induced by the contraction of $R_{\mathbf{q}+1}^1$ and $R_{\mathbf{p}+1}^0$ with $e_{\mathbf{p}}^{\mathbf{Y}_0}$ and $\omega_{\mathbf{q}}^{\mathbf{Y}_1}$, respectively. Since (i) all *dynamical* fields are contained in $e_{\mathbf{p}}^{\mathbf{Y}_0}$, i.e., $\omega_{\mathbf{q}}^{\mathbf{Y}_1}$ is an *auxiliary* field; (ii) $R_{\mathbf{p}+1}^0 = 0$ does not impose any dynamical equations on $e_{\mathbf{p}}^{\mathbf{Y}_0}$ [55], expressing $\omega_{\mathbf{q}}^{\mathbf{Y}_1}$ in terms of the first derivatives of $e_{\mathbf{p}}^{\mathbf{Y}_0}$ modulo σ_- closed terms; it follows that π_1 is trivial in the sense that it allows one to express non-exact part of $\omega_{\mathbf{q}}^{\mathbf{Y}_1}$ via the first derivatives of $e_{\mathbf{p}}^{\mathbf{Y}_0}$. Since $R_{\mathbf{q}+1}^1$ has two indices more than $e_{\mathbf{p}}^{\mathbf{Y}_0}$, π_0 contracts two indices, one form and one tangent, and imposes, if needed, the Young symmetry conditions, which are trivial in symmetric basis. By virtue of (3.10) π_0 sends $\sigma_-(\omega_{\mathbf{r}}^{\mathbf{Y}_2})$ to zero in accordance with the fact that $\omega_{\mathbf{r}}^{\mathbf{Y}_2}$ does not contribute to the dynamical equations, i.e. the term $\langle e_{\mathbf{p}}^{\mathbf{Y}_0} | \sigma_-(\omega_{\mathbf{r}}^{\mathbf{Y}_2}) \rangle$ can in principle be added to the action but it is identically zero.

The action is to be invariant under the standard gauge transformations (2.13)

$$\delta e_{\mathbf{p}}^{\mathbf{Y}_0} = D\xi_{\mathbf{p}-1}^{\mathbf{Y}_0} + \sigma_-(\xi_{\mathbf{q}-1}^{\mathbf{Y}_1}), \quad \xi_{\mathbf{p}-1}^{\mathbf{Y}_0} \in \mathcal{W}_{\mathbf{p}-1}^{g=0}, \quad \xi_{\mathbf{q}-1}^{\mathbf{Y}_1} \in \mathcal{W}_{\mathbf{q}-1}^{g=1}, \quad (3.17)$$

$$\delta \omega_{\mathbf{q}}^{\mathbf{Y}_1} = D\xi_{\mathbf{q}-1}^{\mathbf{Y}_1} + \sigma_-(\xi_{\mathbf{r}-1}^{\mathbf{Y}_2}), \quad \xi_{\mathbf{r}-1}^{\mathbf{Y}_2} \in \mathcal{W}_{\mathbf{r}-1}^{g=2}. \quad (3.18)$$

Despite the fact that $\omega_{\mathbf{r}}^{\mathbf{Y}_2}$ does not contribute to the dynamical equations, the action reveals a symmetry with gauge parameter $\xi_{\mathbf{r}-1}^{\mathbf{Y}_2}$.

To check this invariance it is useful to rewrite the action as $S = \langle R_{\mathbf{p}+1}^0 | \omega_{\mathbf{q}}^{\mathbf{Y}_1} \rangle + \alpha \langle \sigma_-(\omega_{\mathbf{q}}^{\mathbf{Y}_1}) | \omega_{\mathbf{q}}^{\mathbf{Y}_1} \rangle$. The action turns out to be invariant under gauge transformations both with $\xi_{\mathbf{p}-1}^{\mathbf{Y}_0}$ and $\xi_{\mathbf{r}-1}^{\mathbf{Y}_2}$ for any α . However in order to cancel the variation with respect to $\xi_{\mathbf{q}-1}^{\mathbf{Y}_1}$ the value of $\alpha = -\frac{1}{2}$ is required. Indeed,

$$\delta S = \langle R_{\mathbf{p}+1}^0 | \delta \omega_{\mathbf{q}}^{\mathbf{Y}_1} \rangle + 2\alpha \langle \sigma_-(\omega_{\mathbf{q}}^{\mathbf{Y}_1}) | \delta \omega_{\mathbf{q}}^{\mathbf{Y}_1} \rangle = \langle R_{\mathbf{p}+1}^0 + 2\alpha \sigma_-(\omega_{\mathbf{q}}^{\mathbf{Y}_1}) | \delta \omega_{\mathbf{q}}^{\mathbf{Y}_1} \rangle, \quad (3.19)$$

where the symmetry property (3.9) and $\delta R_{\mathbf{p}+1}^0 \equiv 0$ have been used. Then, substituting (3.18)

$$\delta S = \langle R_{\mathbf{p}+1}^0 + 2\alpha \sigma_-(\omega_{\mathbf{q}}^{\mathbf{Y}_1}) | D\xi_{\mathbf{q}-1}^{\mathbf{Y}_1} \rangle + \langle R_{\mathbf{p}+1}^0 + 2\alpha \sigma_-(\omega_{\mathbf{q}}^{\mathbf{Y}_1}) | \sigma_-(\xi_{\mathbf{r}-1}^{\mathbf{Y}_2}) \rangle, \quad (3.20)$$

the second term vanishes due to (3.10). By virtue of $D^2 = 0$ (2.9) and $Dh^a = 0$ (2.8) the d -form in the second term of

$$\delta S = (1 + 2\alpha) \langle \sigma_- (\omega_q^{\mathbf{Y}_1}) | D\xi_{q-1}^{\mathbf{Y}_1} \rangle + \langle De_{\mathbf{p}}^{\mathbf{Y}_0} | D\xi_{q-1}^{\mathbf{Y}_1} \rangle \quad (3.21)$$

is exact and, hence, $\delta S = 0$ provided $\alpha = -\frac{1}{2}$. Consequently, the action becomes

$$S = \langle De_{\mathbf{p}}^{\mathbf{Y}_0} + \frac{1}{2} \sigma_- (\omega_q^{\mathbf{Y}_1}) | \omega_q^{\mathbf{Y}_1} \rangle \quad (3.22)$$

Inasmuch as action (3.22) is gauge invariant, the Lagrangian equations can be written in terms of field strengths (3.5) and (3.6). In Cartesian coordinates the variation of the action reads

$$\begin{aligned} \frac{\delta S}{\delta \omega} \delta \omega &= R_{\mu[p+1]}^{ua(s_1-2), \dots, uc(s_r-2), \overbrace{u, \dots, u}^{q-r}} \delta \omega_{\mu[q]}^u{}_{a(s_1-2) \dots c(s_r-2)} \overbrace{u, \dots, u}^{p+1-r} \delta_{u[p+q+1]}^{\mu[p+q+1]} = 0 \Rightarrow \\ &\Rightarrow R^{ua(s_1-2), \dots, uc(s_r-2), \overbrace{u, \dots, u}^{q-r}} | v[p+1] \delta \omega_{va(s_1-2), \dots, vc(s_r-2), \underbrace{v, \dots, v}_{p+1-r}} | u[q] = 0 \Rightarrow \\ &\Rightarrow R^{a_1(s_1-1), \dots, a_r(s_r-1), a_{r+1}, \dots, a_q | a_1 \dots a_r \dots a_q \dots a_{p+1}} = 0 \end{aligned} \quad (3.23)$$

$$\begin{aligned} \frac{\delta S}{\delta e} \delta e &= R_{\mu[q+1]}^{ua(s_1-2), \dots, uc(s_r-2), \overbrace{u, \dots, u}^{p+1-r}} \delta e_{\mu[p]}^u{}_{a(s_1-2) \dots c(s_r-2)} \overbrace{u, \dots, u}^{q-r} \delta_{u[p+q+1]}^{\mu[p+q+1]} = 0 \Rightarrow \\ &\Rightarrow R^{ua(s_1-2), \dots, uc(s_r-2), \overbrace{u, \dots, u}^{p-r}} | v[q] \delta e_{va(s_1-2), \dots, vc(s_r-2), \underbrace{v, \dots, v}_{q-r}} | u[p] = 0 \Rightarrow \\ &\Rightarrow R^{a_1(s_1-1), \dots, a_r(s_r-1), a_{r+1}, \dots, a_q, \dots, a_p, m | a_1 \dots a_q \dots a_p} = 0 \end{aligned} \quad (3.24)$$

which gives π_1 and π_0 with required properties. Indeed, after fixing the algebraic gauge symmetry for $e_{\mathbf{p}}^{\mathbf{Y}_0}$ the dynamical components of $e_{\mathbf{p}}^{\mathbf{Y}_0}$ are equivalent to the Labastida field $\phi_{\mathbf{Y}}$. Then, by definition of \mathbf{Y}_1 and σ_- , the nonzero components of $De_{\mathbf{p}}^{\mathbf{Y}_0}$ match σ_- -nonclosed components of $\omega_q^{\mathbf{Y}_1}$ (actually all σ_- -closed components of $\omega_q^{\mathbf{Y}_1}$ are σ_- -exact and, hence, are pure gauge by virtue of $\xi_{r-1}^{\mathbf{Y}_2}$ [55]).

Note, that to derive the dynamical equations it is sufficient to set to zero the components of the field strengths $R_{\mathbf{p}+1}^{\mathbf{Y}_0}$ and $R_{\mathbf{q}+1}^{\mathbf{Y}_1}$ with the symmetry of $\mathbf{Y}_d = \mathbf{Y}\{h_1+1, h_2, \dots, h_{s_1}\} \equiv \mathbf{Y}\{(s_1, p_1), \dots, (s_N, p_N), 1\}$ and \mathbf{Y} , respectively (see a spin- $\mathbf{Y}\{s, t\}$ example). It is the projection onto \mathbf{Y}_d and \mathbf{Y} that is made in (3.23) and (3.24). Obviously, π_1 allows one to express the components of $\omega_q^{\mathbf{Y}_1}$ with the symmetry of \mathbf{Y}_d in terms of $De_{\mathbf{p}}^{\mathbf{Y}_0}$ from $\pi_1 [R_{\mathbf{p}+1}^{\mathbf{Y}_0}] = 0$. π_0 is even more trivial and allows one to project $R_{\mathbf{q}+1}^{\mathbf{Y}_1}$ onto components with the symmetry of \mathbf{Y} .

For example, for a spin- \boxplus field action (3.22) has the form

$$S = \int \left[de_2^u + \frac{1}{2} h_b h_c \omega_1^{abc} \right] \omega_1^{uuu} \epsilon_{uuuv(d-4)} h^v \dots h^v, \quad (3.25)$$

and after passing to world indices coincides with [35]. For a spin- $\overline{\boxplus}$ field action (3.22) has the form

$$S = \int \left[de_2^{ua(s-2), ub(t-2)} + \frac{1}{2} h_c \omega_2^{ua(s-2), ub(t-2), c} \right] \omega_2^u{}_{a(s-2)} \omega_2^u{}_{b(t-2)} \epsilon_{u(5)v(d-5)} h^v \dots h^v \quad (3.26)$$

with the Lagrangian equations

$$R^{ua(s-2),ub(t-2)|mmm} \delta \omega_{ma(s-2),mb(t-2),m|uu} = 0 \quad \Rightarrow \quad R^{a(s-1),b(t-1)|abc} = 0, \quad (3.27)$$

$$R^{ua(s-2),ub(t-2),c|mm} \delta e_{ma(s-2),mb(t-2)|uu} = 0 \quad \Rightarrow \quad R^{a(s-1),b(t-1),c|ab} = 0. \quad (3.28)$$

It is from these equations that the dynamical equations have been recovered in the previous section.

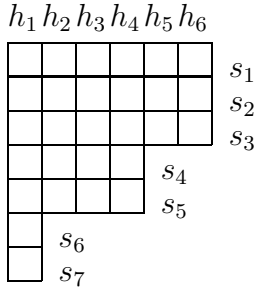
To conclude, we note that despite the fact that the action (3.22) is not built of field strengths and, hence, is not manifestly gauge invariant, equations of motion (3.15-3.16) are written in terms of gauge invariant field strengths. Simplicity of the *frame-like* action (3.22) is very encouraging and we hope it will be very helpful for the further study of mixed-symmetry fields.

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A Young diagrams

When working with mixed-symmetry tensors an essential use is made of Young diagrams. A Young diagram is a picture consisting of n left-justified rows made of boxes, with the i -th row containing s_i boxes, s_i being non-increasing function of i .



There is a number of ways to define a particular Young diagram that are used in applications. A Young diagram can be defined by directly enumerating the lengths of rows $\mathbf{Y}\{s_1, s_2, \dots, s_n\}$, e.g., $\mathbf{Y}\{6, 6, 6, 4, 4, 1, 1\}$; by enumerating the heights of columns $\mathbf{Y}\{h_1, h_2, \dots, h_{s_1}\}$, e.g., $\mathbf{Y}\{7, 5, 5, 5, 3, 3\}$; combining the rows of equal lengths into blocks $\mathbf{Y}\{(s_1, p_1), \dots, (s_N, p_N)\}$, p_i being the number of rows of the length s_i , e.g., $\mathbf{Y}\{(6, 3), (4, 2), (1, 2)\}$.

Each Young diagram defines¹⁹ an irreducible under permutations of indices type of a tensor. There are two main bases that are used for tensors, namely, symmetric and antisymmetric. A tensor $T^{a(s_1), b(s_2), \dots, d(s_n)}$ is said to have the symmetry of $\mathbf{Y}\{s_1, s_2, \dots, s_n\}$, being taken in symmetric basis, iff it is symmetric in each group of indices $a_1 \dots a_{s_1}$, $b_1 \dots b_{s_2}, \dots$, separately, and the symmetrization of all indices from any group with one index from the next group vanishes, i.e.,

¹⁹We do not go into details of this correspondence, for systematic presentation see [60].

$T^{a(s_1),\dots,b(s_i),bc(s_{i+1}-1),\dots,d(s_n)} \equiv 0$. In antisymmetric basis $T^{a[h_1],b[h_2],\dots,d[h_{s_1}]}$ is antisymmetric in each group of indices $a_1\dots a_{h_1}$, $b_1\dots b_{h_2}$, ..., separately, and the antisymmetrization of all indices from any group with one index from the next group vanishes. To make tensors of the Lorentz algebra irreducible in addition to the Young symmetry conditions a traceless condition has to be imposed, i.e., the contraction of the $\mathfrak{so}(d-1,1)$ -invariant metric η_{ab} with any pair of indices must vanish. It is also required $h_1 + h_2 \leq d$ since the Young symmetry together with the tracelessness makes $\mathfrak{so}(d-1,1)$ -tensors with $h_1 + h_2 > d$ be identically zero. We do not consider (anti)-self dual representations of $\mathfrak{so}(d-1,1)$.

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